Counting Unordered Pairs

Three perspectives on $\binom{n}{2}$

Problem statement

Motivating problem

Five schools are sending their baseball teams to a tournament. Each team must play every other team **exactly once**. How many games are required in total?

General form. Given a set $S = \{s_1, s_2, \ldots, s_n\}$ of n distinct elements, how many *unordered pairs* $\{s_i, s_j\}$ with $s_i \neq s_j$ can be formed?

We solve this three ways. Each approach reveals a different combinatorial principle, and all three land on the same answer:

$$\boxed{\frac{n(n-1)}{2} = \binom{n}{2}}$$

Approach 1: The concrete case $n = 5$

Lay out a 5×5 table whose rows and columns are indexed by the teams $T_1, \ldots, T_5$. A check mark in row T_i, column T_j records the game between team i and team j.

	T_1	T_2	T_3	T_4	T_5	*count*
T_1	×	✓	✓	✓	✓	4
T_2	×	×	✓	✓	✓	3
T_3	×	×	×	✓	✓	2
T_4	×	×	×	×	✓	1
T_5	×	×	×	×	×	0

Reading the table.

- **Diagonal** (×, red): a team cannot play itself.
- **Lower triangle** (×, gray): the pair (T_i, T_j) with $i > j$ records the *same* game as (T_j, T_i), so we strike it out to avoid double-counting.
- **Upper triangle** (✓, green): exactly one entry per distinct unordered pair.

Counting the checks row by row gives

$$4 + 3 + 2 + 1 + 0 \;=\; 10 \text{ games.}$$

Approach 2: Row-by-row generalization

The same table works for any n. In row T_k, the cell in column T_j is a check exactly when $j > k$, so row T_k contributes $n - k$ unordered pairs.

Key observation

For row T_k, any element up to and including T_k itself cannot be considered: the diagonal cell (T_k playing itself) is excluded, and all cells with $j < k$ are already accounted for in earlier rows. Only the $n - k$ entries with $j > k$ contribute.

Summing over all rows:

$$\begin{aligned}
\text{Total} \;=\; \sum_{k=1}^{n}(n-k) \;&=\; \underbrace{(n-1) + (n-2) + \cdots + 1 + 0}_{n \text{ terms}} \\
&=\; \sum_{k=1}^{n} n \;-\; \sum_{k=1}^{n} k \\
&=\; n \cdot n \;-\; \frac{n(n+1)}{2} \\
&=\; \frac{2n^2 - n(n+1)}{2} \;=\; \frac{n(n-1)}{2}.
\end{aligned}$$

Sanity check. For $n = 5$: $\dfrac{5 \cdot 4}{2} = 10$. ✓

Approach 3: Count ordered pairs, then divide

Forget the table. Think of building a pair by filling two labeled slots in sequence:

$$(\ \underline{\text{ 1st }}\ ,\ \underline{\text{ 2nd }}\ ).$$

- The first slot can be filled in n ways (any element of S).
- Once the first slot is fixed, the second slot can be filled in $n - 1$ ways: any element *except* the one already chosen (since the pair must contain two different elements).

Therefore the number of **ordered** pairs of distinct elements is

$$n(n-1).$$

From ordered to unordered: the division principle

Each unordered pair $\{s_i, s_j\}$ corresponds to *exactly two* ordered pairs, namely (s_i, s_j) and (s_j, s_i). The count $n(n-1)$ therefore double-counts every unordered pair, and we recover the unordered count by dividing by the size of the symmetry group (2):

$$\text{unordered pairs} \;=\; \frac{n(n-1)}{2}.$$

This is a special case of a general rule: *when every object is counted the same number of times m, divide by m.*

Result

Number of unordered pairs from n distinct elements

$$\binom{n}{2} \;=\; \frac{n(n-1)}{2}.$$

For the baseball tournament with $n = 5$ teams:

$$\binom{5}{2} \;=\; \frac{5 \cdot 4}{2} \;=\; 10 \text{ games}.$$

Cross-check: small values

n	Row-sum $\sum_{k=1}^{n}(n-k)$	Ordered/2 $= n(n-1)/2$	$\binom{n}{2}$
2	$1+0=1$	$2 \cdot 1/2 = 1$	1
3	$2+1+0=3$	$3 \cdot 2/2 = 3$	3
4	$3+2+1+0=6$	$4 \cdot 3/2 = 6$	6
5	$4+3+2+1+0=10$	$5 \cdot 4/2 = 10$	10
6	$5+4+3+2+1+0=15$	$6 \cdot 5/2 = 15$	15

All three columns agree, as they must.

Conclusion

Summary

The number of unordered pairs of distinct elements drawn from an n-element set is

$$\binom{n}{2} \;=\; \frac{n(n-1)}{2}.$$

We proved this three ways:

1. **Concrete table** ($n = 5$): count check marks in the upper triangle.
2. **Row-by-row sum**: row T_k contributes $n-k$ pairs; summing yields $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.
3. **Division principle**: $n(n-1)$ ordered pairs, each unordered pair counted twice, so divide by 2.

The three viewpoints — *enumerate carefully*, *sum a pattern*, *overcount and divide* — are the standard combinatorial moves and recur throughout the subject.

Remark (where this quantity shows up). The number $\binom{n}{2}$ counts:

- the *edges of the complete graph* K_n on n vertices;
- the *handshakes* exchanged when n people each shake hands with every other person once;
- the *number of* 2*-element subsets* of an n-set;
- the n*-th triangular number* $T_{n-1} = 1 + 2 + \cdots + (n-1)$.

Remark (generalization). The third approach extends immediately: the number of unordered r-subsets of an n-set is

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

obtained by counting ordered r-tuples of distinct elements ($n(n-1)\cdots(n-r+1)$ of them) and dividing by the number of orderings of each r-subset ($r!$). For $r = 2$ this recovers $\frac{n(n-1)}{2}$.